Oscillations and Simple Harmonic Motion

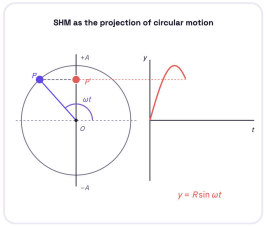
Last-minute revision: the key ideas, standard results and traps of every part. The formula sheet is the last page.

If you only have five minutes

1. SHM: $a = -\omega^2 x$; $y = A\sin(\omega t + \phi)$, direction of motion picks ϕ or $\pi - \phi$.
2. $v = \omega\sqrt{A^2 - y^2}$, $a_{max} = \omega^2 A$; $E = \frac{1}{2}kA^2$, $K = U$ at $A/\sqrt{2}$.
3. Periods: $2\pi\sqrt{l/g_{eff}}$, $2\pi\sqrt{m/k}$, $2\pi\sqrt{I/mgl}$, $2\pi\sqrt{I/C}$, $2\pi\sqrt{\mu/k}$.
4. Reference circle: time = $\Delta\theta/\omega$ ($T/12$ to $A/2$); sudden change: $A' = \sqrt{x^2 + v'^2/\omega'^2}$.
5. Any system: $\omega = \sqrt{k_{eff}/M_{eff}} = \sqrt{U''(x_0)/m}$; constant forces only shift the mean.

1 Periodic Motion, Oscillations and When Motion Is SHM

- v = 1/T; amplitude = maximum displacement from the mean.
- SHM: a = −ω²y, F = −ky, toward the mean.
- Projection of circular motion: y = R sin ωt.



STANDARD RESULTS

R = 5 cm, 3 rev/s: $v_{max} \approx 0.94$ m/s, $a_{max} \approx 17.8$ m/s².

TRAPS

× Zero velocity at the extreme means zero force The restoring force and the acceleration are largest at the extremes.

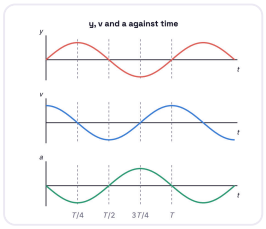
× Every oscillation is SHM A ball bouncing on the floor is periodic but not simple harmonic.

× a = −ω²(x − x₀) is not SHM It is SHM about the mean position x_0.

× The reference particle is the moving body It is a construction; only the projection P′ oscillates.

2 Equation of SHM, Phase, Velocity and Acceleration

- y = A sin(ωt + α); direction picks α or π − α.
- v = ω√(A² − y²): Aω at the mean, 0 at ±A.
- a = −ω²y: 0 at the mean, ω²A at the extremes.



STANDARD RESULTS

A = 0.5 m, T = π/2 s, y = 0.3 m: v = 1.6 m/s, K = 0.64 J (m = 0.5 kg).

TRAPS

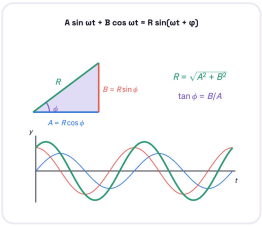
× Acceleration is largest where speed is largest It is the opposite: a = 0 at the mean, where v is largest.

× Measuring y from the end of the path in v = ω√(A² − y²) y is measured from the mean position.

× Fixing α from the position alone The direction of motion picks between α and π − α.

3 Reading SHM Equations and Apparent Weight on an Oscillating Platform

- Test: d²y/dt² = −ω²y.
- A sin ωt + B cos ωt: R = √(A² + B²), tan φ = B/A.
- Platform: N = mg ± mω²A; weightless at the top when ω²A = g.



STANDARD RESULTS

y = 5 cos(20t − π/6) cm, 2 kg: phase π/3, KE_{max} = 1 J.

TRAPS

× Adding A + B for A sin ωt + B cos ωt They are 90° apart: R = √(A² + B²).

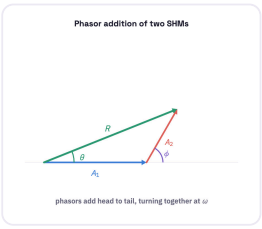
× Reading a cosine's phase as a sine's cos θ = sin(θ + π/2): add π/2.

× Mixing cm and m in joules Convert the amplitude to metres first.

× Largest scale reading at the top It is at the lowest point, where a is upward.

4 Superposition of SHMs: Same and Different Frequencies

- Same ω: R = √($A_1^2 + A_2^2$ + $2A_1A_2$ cos φ).
- Different ω: not one SHM; count terms after expanding.
- Constants shift the mean only.



STANDARD RESULTS

5 sin ωt + 3 sin(ωt + 60°): R = 7, θ ≈ 21.8°.

TRAPS

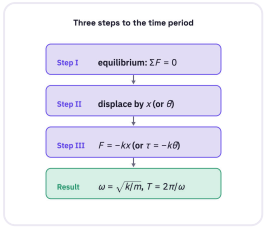
× Adding amplitudes $A_1 + A_2$ for SHMs out of phase Add them as vectors at the phase difference φ.

× Counting the factors of a product Expand into a sum and count the terms.

× Counting a constant as an SHM A constant only shifts the mean position.

5 Angular SHM and the Three-Step Method to Find the Time Period

- $\theta = \theta_0 \sin(\omega t + \alpha)$, $d^2\theta/dt^2 = -\omega^2\theta$.
- $\tau = -I\omega^2\theta$; $\omega = \sqrt{(k/I)}$ for $\tau = -k\theta$.
- Method: equilibrium → displace → $F = -kx$ → $\omega = \sqrt{(k/m)}$.



STANDARD RESULTS

0.5 kg between 30 and 20 N/m: $T = \pi/5$ s.

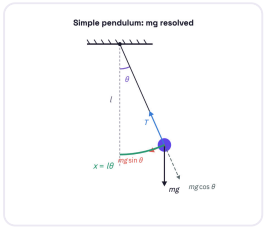
TRAPS

✕ **Taking the angular velocity dθ/dt as ω** dθ/dt changes during the swing; ω is the constant angular frequency.

✕ **Keeping constant forces in the restoring force** Measure the displacement from equilibrium; balanced forces cancel.

6 Simple Pendulum and Spring-Block Pendulum

- $T = 2\pi\sqrt{(l/g)}$: no mass; small swings only.
- $T_{min} = mg \cos\theta_0$, $T_{max} = mg(3 - 2\cos\theta_0)$.
- Spring-block: $T = 2\pi\sqrt{(m/k)} = 2\pi\sqrt{(h/g)}$.



STANDARD RESULTS

$T_{max} = 1.5\, T_{min} \Rightarrow \cos\theta_0 = 6/7$, $\theta_0 \approx 31°$.

TRAPS

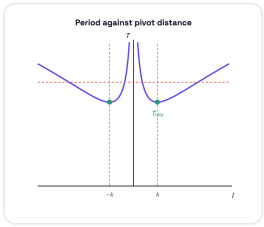
✕ **The period depends on the bob's mass** $T = 2\pi\sqrt{(l/g)}$ has no m.

✕ **Using $2\pi\sqrt{(l/g)}$ for large swings** It needs $\sin\theta \approx \theta$ (small angles).

✕ **g changes the period of a spring-block** g only shifts the equilibrium; $T = 2\pi\sqrt{(m/k)}$.

7 Compound and Torsional Pendulums, Equivalent Length and Minimum Period

- $T = 2\pi\sqrt{(I/mgl)}$, I about the pivot.
- Torsional: $T = 2\pi\sqrt{(I/C)}$, no g.
- $l_{eq} = l + k^2/l$; $T_{min} = 2\pi\sqrt{(2k/g)}$ at $l = k$.



STANDARD RESULTS

1.2 m rod: $x = 0.35$ m, $T_{min} \approx 1.67$ s.

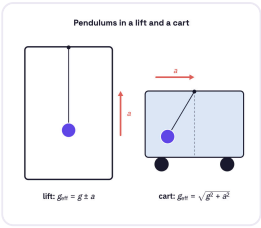
TRAPS

✕ **Using I about the centre of mass** I is about the pivot: $I_C + ml^2$.

✕ **Assuming the period keeps falling as the pivot nears C** It is smallest at $l = k$ and rises again inside.

8 Time Period in Physical Situations: Strings, Effective g and Liquids

- String: $T = 2\pi\sqrt{(ml/4T)}$.
- $T = 2\pi\sqrt{(l/g_{eff})}$: lift $g \pm a$, cart $\sqrt{(g^2 + a^2)}$, liquid $g(1 - \sigma/\rho)$.
- Float: $2\pi\sqrt{(m/A\rho g)}$; U-tube: $2\pi\sqrt{(L/2g)}$.



STANDARD RESULTS

10 g, +5 μC, $E = 10^4$ N/C up: $T \approx 2.81$ s (1.62 s reversed).

TRAPS

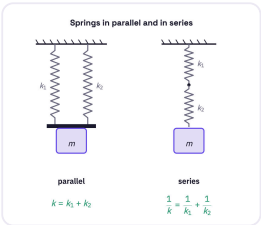
✕ **Using g + a for a sideways-accelerating cart** They are perpendicular: $g_{eff} = \sqrt{(g^2 + a^2)}$.

✕ **Forgetting buoyancy for a bob in a liquid** $g_{eff} = g(1 - \sigma/\rho)$.

✕ **Taking the U-tube level difference as y** Pushing one side down by y raises the other by y: 2y.

9 Springs in Series and Parallel, Cut Springs, Pulleys and Rods on Springs

- Parallel $k_1 + k_2$; series $1/k_1 + 1/k_2$, $T^2 = t_1^2 + t_2^2$.
- Cut: kl constant; n parts → nk each.
- Rod on springs: $\omega = \sqrt{((12kl + 3mg)/(2ml))}$.



STANDARD RESULTS

Movable pulley, 2 kg, 800 N/m: $k_{eff} = 4k$, $T \approx 0.157$ s.

TRAPS

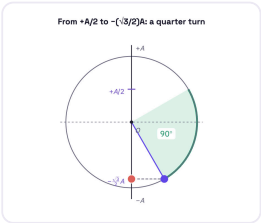
✕ **Two springs on opposite sides of a block taken as series** Both push it back: parallel, $k_1 + k_2$.

✕ **Spring stretch = block displacement with a pulley** Work out the stretch from the string geometry.

✕ **Dropping gravity's torque on a hanging rod** It adds $mg(l/2)\theta$ (and opposes for an inverted rod).

10 Phase Difference and Time Between Two Positions in SHM

- Same ω: Δφ constant; different ω: $(\omega_1 - \omega_2)t$.
- $\sin\varphi = y/A$; direction picks φ or $\pi - \varphi$.
- $t = \Delta\theta/\omega$: T/12 to A/2, T/8 to A/$\sqrt{2}$, T/4 to the extreme.



STANDARD RESULTS

$A = 10$ cm, $T = 12$ s, $+5 \to -5\sqrt{3}$ cm: 3 s.

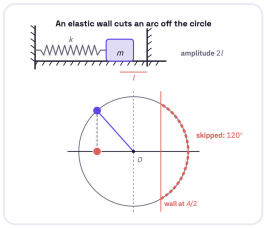
TRAPS

✕ **Using $\sin^{-1}(y/A)$ for both particles** The one moving the other way is in the other quadrant: $\pi - \sin^{-1}(y/A)$.

✕ **Dividing distance by an average speed** Speed varies in SHM; use the angle swept on the reference circle.

11 Sudden Changes in SHM: Collisions, Added Mass and Elastic Walls

- Momentum conserved in the instant; new ω and new mean.
- $A' = \sqrt{(x^2 + v'^2/\omega'^2)}$; at the mean $A\sqrt{(m/(m + m'))}$.
- Wall: $T = \sqrt{(m/k)}[\pi + 2\sin^{-1}(l/A)]$.



STANDARD RESULTS

3 kg + 1 kg at the mean, A = 20 cm: $A' \approx 17.3$ cm.

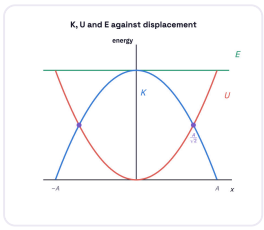
TRAPS

✕ **Keeping the old ω after mass is added** $\omega' = \sqrt{(k/(m + m'))}$.

✕ **Measuring position from the old equilibrium of a vertical spring** The added mass lowers the equilibrium by m′g/k.

12 Energy in SHM: Potential Well, Kinetic and Potential Energy

- $F = -dU/dx$; small oscillations: $k = U''(x_0)$.
- $K = \frac{1}{2}m\omega^2(A^2 - x^2)$, $U = \frac{1}{2}m\omega^2x^2$, $E = \frac{1}{2}kA^2$.
- $K = U$ at $A/\sqrt{2}$; K, U at 2ω, average E/2.



STANDARD RESULTS

0.5 kg, ω = 20, K = 0.16 J at 3 cm: A = 5 cm.

TRAPS

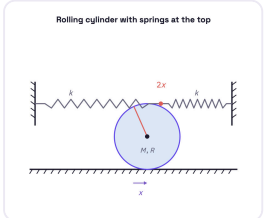
✕ **Taking the origin as the mean when U(x) has a linear term** Solve $dU/dx = 0$ for x_0 first.

✕ **Measuring x from the natural length on a vertical spring** Measure from the equilibrium; gravity then drops out.

✕ **K and U oscillate at the SHM frequency** They oscillate at 2ω (period T/2).

13 Energy Method to Find the Frequency: Pulleys and Rolling Bodies

- $dE_T/dt = 0$, or $\omega = \sqrt{(k_{eff}/M_{eff})}$.
- Pulley: + I/r^2; loop pulley: $k_{eff} = 4k$.
- Rolling cylinder: $M_{eff} = 3M/2$; top springs stretch 2x.



STANDARD RESULTS

2 kg, $I/r^2 = 2$ kg, k = 400: $T = \pi/5$ s.

TRAPS

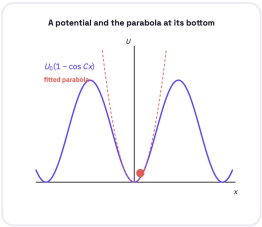
✕ **Leaving rotational KE out of the total energy** Add $\frac{1}{2}I\omega_{rot}^2$ for every turning pulley or rolling body.

✕ **A pulley moving x stretches the spring x** In a loop of string it stretches 2x: $k_{eff} = 4k$.

✕ **A spring on the top of a rolling rim deforms x** The top moves 2x.

14 SHM in a Given Potential and Mean Kinetic Energy

- $F = -dU/dx$; $k_{eff} = U''$ at the minimum.
- $U_0(1 - \cos Cx)$: $T = 2\pi\sqrt{(m/U_0C^2)}$.
- $\langle K \rangle = \langle U \rangle = E/2$ over a period.



STANDARD RESULTS

$U_0 \sin^2(Cx)$, 0.2 kg, $U_0 = 5$ J, C = 2: $T \approx 0.44$ s.

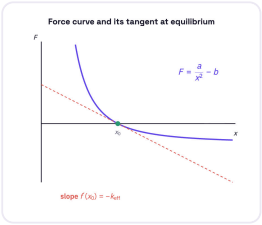
TRAPS

✕ **Using the coefficient of x^2 as k** For $U = ax^2$, $F = -2ax$: $k = U'' = 2a$.

✕ **Taking the mean KE as the maximum KE** Over a period it is half: E/2.

15 Taylor's Method: ω From the Force or Potential Function

- $\omega = \sqrt{(|f'(x_0)|/m)} = \sqrt{(U''(x_0)/m)}$.
- −q between two +Q: $k = 2k_{eQq}/r^3$; ring axis: k_{eQq}/R^3.
- Charged spring doubling to 2l: $k_{eff} = 2k$.



STANDARD RESULTS

$U = 4/x + x$, 0.5 kg: $T = \pi\sqrt{2} \approx 4.44$ s.

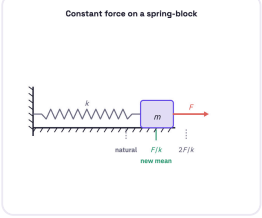
TRAPS

✕ **Evaluating f′ at x = 0 when the equilibrium is elsewhere** Expand about x_0, where $f(x_0) = 0$.

✕ **Mixing the Coulomb constant with a spring constant** Write the Coulomb constant $k_e = 1/4\pi\varepsilon_0$.

16 SHM About a Shifted Mean Position: Constant Forces and Lifts

- $\ddot{y} + \omega^2 y = C$: SHM about C/ω^2.
- Constant F: mean at F/k, max extension 2F/k, ω unchanged.
- Lift: amplitude ma/k.



STANDARD RESULTS

$f = -20x + 4$, 0.2 kg, from rest at 1 m: $x = 0.2 + 0.8\cos 10t$.

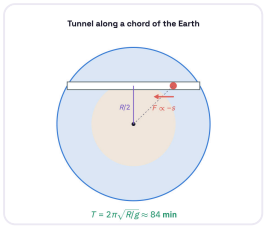
TRAPS

✕ **Taking the release point's distance from the origin as the amplitude** Measure from the mean position x_m.

✕ **A constant force changes the period** It only moves the mean; $\omega = \sqrt{(k/m)}$.

17 Two-Body Oscillations, the Bifilar Rod and the Tunnel Through the Earth

- Two free bodies: $T = 2\pi\sqrt{(\mu/k)}$, $\mu = m_1m_2/(m_1 + m_2)$.
- Bifilar rod: $T = 2\pi\sqrt{(l/3g)}$.
- Earth tunnel, any chord: $T = 2\pi\sqrt{(R/g)} \approx 84$ min.



STANDARD RESULTS

A 1 kg, B 4 kg, 200 N/m, A at 5 m/s: $x_A = t + 0.253 \sin 15.8t$.

TRAPS

✕ **Using one block's mass when the other is free** Use the reduced mass $\mu = m_1m_2/(m_1 + m_2)$.

✕ **Using GM/x^2 inside the Earth** Only the mass inside radius x pulls: $F \propto x$.

Oscillations and Simple Harmonic Motion

g 9.8 m/s² (or 10, π^2 where stated) | k_e $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ N m²C⁻² | R_e 6400 km

1 Periodic Motion, Oscillations and When Motion Is SHM

$v = \frac{1}{T}$

$a = -\omega^2 y$

$F = -ky$

$a = -\omega^2 y$

2 Equation of SHM, Phase, Velocity and Acceleration

$y = A\sin(\omega t + \alpha)$

$T = \frac{2\pi}{\omega}$

$v = \omega\sqrt{A^2 - y^2}$

$v_{max} = A\omega$

$a = -\omega^2 y$

$a_{max} = \omega^2 A$

3 Reading SHM Equations and Apparent Weight on an Oscillating Platform

$A\sin\omega t + B\cos\omega t = R\sin(\omega t + \phi)$

$R = \sqrt{A^2 + B^2}$

$\tan\phi = \frac{B}{A}$

$K_{max} = \frac{1}{2}m\omega^2 A^2$

$\bar{v} = \frac{2v_{max}}{\pi}$

4 Superposition of SHMs: Same and Different Frequencies

$R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$

$\tan\theta = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$

5 Angular SHM and the Three-Step Method to Find the Time Period

$\dot{\theta} = \omega\sqrt{\theta_0^2 - \theta^2}$

$\ddot{\theta} = -\omega^2\theta$

$\tau = -I\omega^2\theta$

$F_R = -kx \Rightarrow \omega = \sqrt{k/m}$

$\tau_R = -k\theta \Rightarrow \omega = \sqrt{k/I}$

$T = \frac{2\pi}{\omega}$

6 Simple Pendulum and Spring-Block Pendulum

$T = 2\pi\sqrt{\frac{l}{g}}$

$T_{max} = mg(3 - 2\cos\theta_0)$

$T_{min} = mg\cos\theta_0$

$\omega = \sqrt{\frac{k}{m}}$

$T = 2\pi\sqrt{\frac{m}{k}}$

7 Compound and Torsional Pendulums, Equivalent Length and Minimum Period

$T = 2\pi\sqrt{\frac{I}{mgl}}$

$\omega = \sqrt{\frac{C}{I}}$

$T = 2\pi\sqrt{\frac{I}{C}}$

$l_{eq} = l_c + \frac{k^2}{l_c}$

$T = 2\pi\sqrt{\frac{l_c + k^2/l_c}{g}}$

8 Time Period in Physical Situations: Strings, Effective g and Liquids

$F_R = -\frac{4T}{l}x$

$T = 2\pi\sqrt{\frac{l}{g - qE/m}}$

$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$

$g_{eff} = \sqrt{g^2 + a^2}$

$T = 2\pi\sqrt{\frac{m}{A\rho g}}$

$T = 2\pi\sqrt{\frac{L}{2g}}$

9 Springs in Series and Parallel, Cut Springs, Pulleys and Rods on Springs

$k_{\parallel} = k_1 + k_2$

$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$

$k \propto \frac{1}{l}$ (cut spring)

$\tau_R = -\left(2kl^2 + \frac{mgl}{2}\right)\theta$

$\omega = \sqrt{\frac{12kl + 3mg}{2ml}}$

10 Phase Difference and Time Between Two Positions in SHM

$\phi = (\omega_1 - \omega_2)t$

$\phi = \sin^{-1}\frac{y}{A}$ or $\pi - \sin^{-1}\frac{y}{A}$

$d_{max} = 2A\sin\frac{\Delta\phi}{2}$

$t = \frac{\Delta\theta}{\omega}$

11 Sudden Changes in SHM: Collisions, Added Mass and Elastic Walls

$A = \sqrt{l^2\alpha^2 + \frac{v^2 l}{g}}$

$\Delta\phi = \sin^{-1}\frac{l\beta}{A_2} + \sin^{-1}\frac{l\alpha}{A_1} - \pi$

$v' = \frac{mv}{m+m'}$

$A' = \sqrt{x^2 + \frac{v'^2}{\omega'^2}}$

$A' = A\sqrt{\frac{m}{m+m'}}$ (at the mean)

$T = \sqrt{\frac{m}{k}}\left[\pi + 2\sin^{-1}\frac{l}{A}\right]$

$T = 2\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \sin^{-1}\frac{\alpha}{\beta}\right]$

12 Energy in SHM: Potential Well, Kinetic and Potential Energy

$F = -\frac{dU}{dx}$

$K = \frac{1}{2}m\omega^2(A^2 - x^2)$

$E = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}kA^2$

$K = U$ at $x = \frac{A}{\sqrt{2}}$

$E = \frac{1}{2}I\omega^2\theta_0^2$

13 Energy Method to Find the Frequency: Pulleys and Rolling Bodies

$\omega = \sqrt{\frac{k_{eff}}{M_{eff}}}$

$T = 2\pi\sqrt{\frac{m + I/r^2}{k}}$

$T = 2\pi\sqrt{\frac{M + I/R^2}{4k}}$

$T = \frac{\pi}{2}\sqrt{\frac{3M}{k}}$

14 SHM in a Given Potential and Mean Kinetic Energy

$T = 2\pi\sqrt{\frac{m}{U_0 C^2}}$

$T = 2\pi\sqrt{\frac{R(M + m + I/R^2)}{mg\cos\alpha}}$

$\langle K\rangle = \frac{1}{4}I\omega^2\beta^2$

$\langle K\rangle = \frac{1}{8}mgl\theta_0^2 + \frac{1}{12}ml^2\dot{\theta}_0^2$

15 Taylor's Method: ω From the Force or Potential Function

$\omega = \sqrt{\frac{f'(x_0)}{m}}$

$\omega = \sqrt{\frac{U''(x_0)}{m}}$

$T = 2\pi\sqrt{\frac{mr^3}{2k_e Qq}}$

$T = 2\pi\sqrt{\frac{m}{2k}}$

16 SHM About a Shifted Mean Position: Constant Forces and Lifts

$y_m = \frac{C}{\omega^2}$

$y = l + \frac{F}{k}(1 - \cos\omega t)$

$v_{max} = \frac{F}{\sqrt{mk}}$

$A = \frac{ma}{k}$ (lift)

$\omega = \sqrt{\frac{k}{m} + \frac{2k_e q_1 q_2}{my_0^3}}$

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$\mu = \frac{m_1 m_2}{m_1 + m_2}$

$T = 2\pi\sqrt{\frac{\mu}{k}}$

$T = 2\pi\sqrt{\frac{l}{3g}}$

$g_{in} = \frac{GM_e x}{R_e^3}$

$T = 2\pi\sqrt{\frac{R_e^3}{GM_e}} = 2\pi\sqrt{\frac{R_e}{g}}$